JOURNAL OF
www.elsevier.com/locate/jalgebra

Engel subgroups of triangular matrices over local rings ☆

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Received 22 January 2004

Available online 24 June 2005

Communicated by Efim Zelmanov

Abstract

We consider Engel subgroups of the group $T(n, R)$ of upper triangular matrices over a local ring R which satisfies a weak commutativity condition. If, in addition, R is artinian then we give a complete description of the maximal Engel subgroups of $T(n, R)$ up to conjugacy. These subgroups turn out to be nilpotent and we study their nilpotency class.

Keywords: Triangular matrix group; Nilpotent subgroup; Engel subgroup

1. Introduction

The group $UT(n, K)$ of the upper-unitriangular $n \times n$-matrices over a field K is a classical example of a nilpotent group which appears naturally in various situations in algebra and geometry. Its direct product with the group of scalar matrices $\{\alpha I\colon 0 \neq \alpha \in K\}$, which we shall denote by $N(n, K)$, gives a maximal nilpotent subgroup of the group $T(n, K)$

☆ This work was partially supported by CNPq and FAPESP of Brazil.
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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.06.029

of the invertible upper-triangular $n \times n$-matrices over K. It is even maximal Engel. It is easily seen that the group of invertible diagonal matrices, which is obviously abelian, is also maximal Engel. What are the other maximal Engel subgroups in $T(n, K)$? Observe that the group of diagonal matrices is the direct product of n copies of $N(1, K) \cong K^*$.

The main result in this paper (Theorem 3.1) implies that an arbitrary maximal Engel subgroup of $T(n, K)$, up to conjugacy in the general linear group $GL(n, K)$, is a direct product of the form $N(n_1, K) \times N(n_2, K) \times \cdots \times N(n_s, K)$ where $n = n_1 + \cdots + n_s$. As a consequence, it follows that it is actually nilpotent. Many interesting results about nilpotent linear groups appear in [1,3,4].

We shall use the following notation: given a ring R we write $e_{ij}(r)$, $r \in R$, for the elementary matrix, whose (i, j)th entry is r and all other entries are 0; $T(n, R)$ denotes the group of invertible upper-triangular $n \times n$-matrices over R, while $UT(n, R)$ stands for the group of upper unitriangular $n \times n$-matrices, that is the subgroup of $T(n, R)$, generated by the upper-triangular transvections $I + e_{ij}(r)$, $i < j$, $r \in R$. We shall also write

$$g = (g_{ij})_{i \leqslant j} = \begin{pmatrix} g_{11} & g_{12} & \dots & g_{1n} \\ 0 & g_{22} & \dots & g_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & g_{nn} \end{pmatrix}.$$

In Section 2 some preliminary results are given. In Section 3 we prove that an Engel subgroup of upper-triangular matrices over a commutative local ring R is conjugate, in $UT(n, R)$, to a group G with the following property:

$$\begin{aligned} &\text{If for a pair of indices } i < j \text{ there is an element } g' = (g'_{ij})_{i \leqslant j} \\ &\text{in } G \text{ such that } g'_{ii} - g'_{jj} \in \mathcal{U}(R), \text{ then } g_{ij} = 0 \text{ for all } g \in G. \end{aligned} \tag{1}$$

Furthermore, with the additional assumption that R is artinian, we describe, up to conjugacy, the maximal Engel subgroups of $T(n, R)$. Moreover, in the latter result we are able to substitute the commutativity condition on a local ring R with maximal ideal $\mathcal{M}$ by the weaker assumption:

$$R/\mathcal{M} \text{ is commutative} \quad \text{and} \quad \mathcal{M}/\mathcal{M}^2 \subseteq \mathfrak{Z}\big(R/\mathcal{M}^2\big), \tag{2}$$

where $\mathfrak{Z}(R/\mathcal{M}^2)$ denotes the centre of $R/\mathcal{M}^2$.

Notice that, for arbitrary $r \in R$ and $m_1, m_2 \in \mathcal{M}$ we have that

$$\begin{aligned} [r, m_1 m_2]_{\mathrm{Lie}} &= r m_1 m_2 - m_1 m_2 r = r m_1 m_2 - m_1\big(r m_2 + [m_2, r]_{\mathrm{Lie}}\big) \\ &= (r m_1 - m_1 r) m_2 - m_1 [m_2, r]_{\mathrm{Lie}} \in \mathcal{M}^3. \end{aligned}$$

Thus $\mathcal{M}^2/\mathcal{M}^3 \subseteq \mathfrak{Z}(R/\mathcal{M}^3)$ and an inductive argument readily shows that condition (2) is actually equivalent to the following:

$$\mathcal{M}^k/\mathcal{M}^{k+1} \subseteq \mathfrak{Z}\big(R/\mathcal{M}^{k+1}\big), \quad \forall k \geqslant 0. \tag{3}$$

A non-commutative example of a local ring satisfying condition (2) is given by the group algebra of a finite p-group G over a field K of characteristic p. In fact, it is well known that this is a local ring whose radical is $\Delta(G)$, the augmentation ideal of KG (see [2, p. 27]). Moreover, in any group ring KG we have that

$$\Delta(G)^k/\Delta(G)^{k+1} \subseteq \mathfrak{Z}\big(KG/\Delta(G)^{k+1}\big).$$

2. Preliminary results

Lemma 2.1. *Suppose that R is either commutative or an artinian local ring satisfying condition* (2)*. Given an upper-triangular $n \times n$-matrix $a = (a_{ij})_{i \leqslant j}$ over R, there is an element $t \in UT(n, R)$ such that for each $1 \leqslant i < j \leqslant n$ with $a_{ii} - a_{jj} \in \mathcal{U}(R)$ the (i, j)th entry of $t^{-1}at$ is 0.*

Proof. For a transvection $t = I + e_{ij}(r)$, $i < j$, $r \in R$, we see that $t^{-1}at$ can be obtained from a by adding to the jth column the ith multiplied by r from the right and substructing from the ith row the jth one multiplied by r from the left. Thus the (i, j)th entry of $t^{-1}at$ is $a_{ij} + a_{ii}r - ra_{jj}$ and the diagonal entries do not change.

Suppose first that R is commutative. Then taking $r = -a_{ij}(a_{ii} - a_{jj})^{-1}$, we see that the (i, j)th entry of $t^{-1}at$ becomes 0. We want to do this for each (i, j) with $a_{ii} - a_{jj} \in \mathcal{U}(R)$ preserving the zeros obtained in previous steps. It is easily seen that this happens if we work parallel to the main diagonal as follows:

$$\begin{aligned}(1,2) &\to (2,3) \to \cdots \to (n-1,n) \to (1,3) \to (2,4) \to \cdots \to (n-2,n)\\ &\to \cdots \to (1,n),\end{aligned}$$

omitting, of course, those places (i, j) for which $a_{ii} - a_{jj} \notin \mathcal{U}(R)$. Thus we obtain an element $t \in UT(n, R)$ such that $t^{-1}at$ possesses the desired property.

Suppose now that R is an artinian local ring with maximal ideal $\mathcal{M}$ satisfying (2). Then $R/\mathcal{M}$ is a field and it follows, by the commutative case, that there is $t_1 \in UT(n, R)$ such that the desired property holds for $t_1^{-1}at_1$ modulo $\mathcal{M}$. Assume by induction that we found already an element $t_k \in UT(n, R)$ such that each (i, j)th entry $a_{ij}^{(k)}$ of $a^{(k)} = t_k^{-1}at_k$ lies in $\mathcal{M}^k$ for every $1 \leqslant i \leqslant j \leqslant n$ with $a_{ii} - a_{jj} \in \mathcal{U}(R)$. Because the element $r = -a_{ij}^{(k)}(a_{ii} - a_{jj})^{-1} \in \mathcal{M}^k$ is central modulo $\mathcal{M}^{k+1}$, we see that r satisfies $a_{ij}^{(k)} + a_{ii}r - ra_{jj} \equiv 0 \bmod \mathcal{M}^{k+1}$. Hence the (i, j)th entry of $(I + e_{ij}(r))^{-1}a^{(k)}(I + e_{ij}(r))$ is contained in $\mathcal{M}^{k+1}$. Working parallel to the main diagonal as we did in the commutative case, we come to an element $t_{k+1} \in UT(n, R)$ such that the (i, j)th entry of $t_{k+1}^{-1}at_{k+1}$ belongs to $\mathcal{M}^{k+1}$ for all $1 \leqslant i \leqslant j \leqslant n$ with $a_{ii} - a_{jj} \in \mathcal{U}(R)$. Since R is artinian, $\mathcal{M}^s = 0$ for some $s \geqslant 0$ and consequently $t_s^{-1}at_s$ satisfies the required property. □

Remark 2.2. Observe that conjugation by a transvection t as above does not change the diagonal entries of a.

Lemma 2.3. *Suppose that R is either commutative or a local ring with condition* (2) *such that $\bigcap_{l=0}^{\infty} \mathcal{M}^l = 0$. Suppose further that an Engel group G of upper-triangular matrices over R contains an element $g = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ in which*

$$a = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_k \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 & & * \\ & \ddots & \\ 0 & & b_m \end{pmatrix}, \quad \text{and} \quad a_i - b_j \in \mathcal{U}(R)$$

for each $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$. Then any $x \in G$ is of the form

$$x = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{with } \alpha = \begin{pmatrix} \alpha_1 & & * \\ & \ddots & \\ 0 & & \alpha_k \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \beta_1 & & * \\ & \ddots & \\ 0 & & \beta_m \end{pmatrix}.$$

Proof. Write $x = \left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \beta \end{smallmatrix}\right)$ in which α and β are as above and $\gamma = (\gamma_{ij})$ is a $k \times m$-matrix. We need to show that $\gamma = 0$.

We have that

$$x^{-1} = \begin{pmatrix} \alpha^{-1} & -\alpha^{-1}\gamma\beta^{-1} \\ 0 & \beta^{-1} \end{pmatrix}$$

and

$$x' = [x, g] = \begin{pmatrix} \alpha^{-1}a^{-1}\alpha a & \alpha^{-1}a^{-1}\gamma b - \alpha^{-1}\gamma\beta^{-1}b^{-1}\beta b \\ 0 & \beta^{-1}b^{-1}\beta b \end{pmatrix}.$$

Compute first $\alpha^{-1}a^{-1}\gamma b$. It equals

$$\begin{pmatrix} \alpha_1^{-1}a_1^{-1} & & & * \\ & \alpha_2^{-1}a_2^{-1} & & \\ & & \ddots & \\ 0 & & & \alpha_k^{-1}a_k^{-1} \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{k1} & \gamma_{k2} & \cdots & \gamma_{km} \end{pmatrix} \begin{pmatrix} b_1 & & & * \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_m \end{pmatrix},$$

which belongs to

$$\begin{pmatrix} \alpha_1^{-1}a_1^{-1}\gamma_{11} + \sum_{i\geqslant 2} R\gamma_{i1} & \alpha_1^{-1}a_1^{-1}\gamma_{12} + \sum_{i\geqslant 2} R\gamma_{i2} & \cdots & \alpha_1^{-1}a_1^{-1}\gamma_{1m} + \sum_{i\geqslant 2} R\gamma_{im} \\ \alpha_2^{-1}a_2^{-1}\gamma_{21} + \sum_{i\geqslant 3} R\gamma_{i1} & \alpha_2^{-1}a_2^{-1}\gamma_{22} + \sum_{i\geqslant 3} R\gamma_{i2} & \cdots & \alpha_2^{-1}a_2^{-1}\gamma_{2m} + \sum_{i\geqslant 3} R\gamma_{im} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_k^{-1}a_k^{-1}\gamma_{k1} & \alpha_k^{-1}a_k^{-1}\gamma_{k2} & \cdots & \alpha_k^{-1}a_k^{-1}\gamma_{km} \end{pmatrix}$$

$$\times \begin{pmatrix} b_1 & & & * \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_m \end{pmatrix}.$$

Similarly, $\alpha^{-1}\gamma\beta^{-1}b^{-1}\beta b$ is contained in

$$\begin{pmatrix} \alpha_1^{-1}\gamma_{11}+\sum_{i\geqslant 2}R\gamma_{i1} & \alpha_1^{-1}\gamma_{12}+\sum_{i\geqslant 2}R\gamma_{i2} & \dots & \alpha_1^{-1}\gamma_{1m}+\sum_{i\geqslant 2}R\gamma_{im} \\ \alpha_2^{-1}\gamma_{21}+\sum_{i\geqslant 3}R\gamma_{i1} & \alpha_2^{-1}\gamma_{22}+\sum_{i\geqslant 3}R\gamma_{i2} & \dots & \alpha_2^{-1}\gamma_{2m}+\sum_{i\geqslant 3}R\gamma_{im} \\ \dots & \dots & \dots & \dots \\ \alpha_k^{-1}\gamma_{k1} & \alpha_k^{-1}\gamma_{k2} & \dots & \alpha_k^{-1}\gamma_{km} \end{pmatrix}$$
$$\times \beta^{-1}b^{-1}\beta b.$$

Suppose now that R is commutative. Then

$$\beta^{-1}b^{-1}\beta b = \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

We see that the $(k,1)$th entry of $\gamma^{(1)} = \alpha^{-1}a^{-1}\gamma b - \alpha^{-1}\gamma\beta^{-1}b^{-1}\beta b$ is $\alpha_k^{-1}\gamma_{k1} \times (a_k^{-1}b_1 - 1)$.

Write

$$x^{(l)} = [x, \underbrace{g,\dots,g}_{l}] = \begin{pmatrix} \alpha^{(l)} & \gamma^{(l)} \\ 0 & \beta^{(l)} \end{pmatrix}, \qquad \gamma^{(l)} = \begin{pmatrix} \gamma_{11}^{(l)} & \gamma_{12}^{(l)} & \dots & \gamma_{1m}^{(l)} \\ \dots & \dots & \dots & \dots \\ \gamma_{k1}^{(l)} & \gamma_{k2}^{(l)} & \dots & \gamma_{km}^{(l)} \end{pmatrix}.$$

Since R is commutative,

$$\alpha^{(l)} = \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \quad \text{and} \quad \beta^{(l)} = \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

The above calculations with $x = x^{(l-1)}$ tell us that $\gamma_{k1}^{(l)} = \alpha_k^{-1}\gamma_{k1}(a_k^{-1}b_1 - 1)^l$ for each $l \geqslant 1$. Since α_k^{-1} and $a_k^{-1}b_1 - 1$ are units in R and G is Engel, it follows that γ_{k1} must be 0. Hence $\gamma_{k1}^{(l)} = 0$ for all $l \geqslant 1$. Then

$$\gamma_{k-1,1}^{(1)} = \alpha_{k-1}^{-1}\gamma_{k-1,1}\big(a_{k-1}^{-1}b_1 - 1\big)$$

and, more generally,

$$\gamma_{k-1,1}^{(l)} = \alpha_{k-1}^{-1}\gamma_{k-1,1}\big(a_{k-1}^{-1}b_1 - 1\big)^l, \quad l \geqslant 1.$$

We derive again from the Engel property of G that $\gamma_{k-1,1} = 0$ and, consequently, $\gamma_{k-1,1}^{(l)} = 0$ for all $l \geqslant 1$. Going up this way by the first column we conclude that $\gamma_{11}^{(l)} =$

$\gamma_{21}^{(l)} = \cdots = \gamma_{k1}^{(l)} = 0$ for each $l \geqslant 1$. Suppose by induction that the first s columns of $\gamma^{(l)}$ are all zero for each $l \geqslant 1$. Then $\gamma_{k,s+1}^{(l)} = \alpha_k^{-1}\gamma_{k,s+1}(a_k^{-1}b_{s+1} - 1)^l$. As above, this yields $\gamma_{k,s+1}^{(l)} = 0$ for every $l \geqslant 1$. It is easily seen that we can go up by the $(s+1)$st column of γ as we did for the first one and conclude that it is also zero. Hence the $(s+1)$st column of $\gamma^{(l)}$ is zero for each $l \geqslant 1$. It follows by induction that γ is the zero $k \times m$-matrix.

Next we adjust our proof for the case when R is a local ring with $\bigcap_{r=0}^{\infty} \mathcal{M}^r = 0$ and which satisfies (2). We keep the above notation. Because $R/\mathcal{M}$ is a field, it follows by the commutative case that all entries of γ are in $\mathcal{M}$. Suppose by induction that we know already that each entry of γ is in $\mathcal{M}^r$. Thus every γ_{ij} is central modulo $\mathcal{M}^{r+1}$. Since working modulo $\mathcal{M}$ we have

$$\beta^{-1}b^{-1}\beta b \equiv \begin{pmatrix} 1 & & * \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix},$$

it follows that $\gamma_{k1}^{(1)} \equiv \alpha_k^{-1}\gamma_{k1}(a_k^{-1}b_1 - 1) \bmod \mathcal{M}^{r+1}$. Furthermore,

$$\alpha^{(l)} \equiv \begin{pmatrix} 1 & & * \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \bmod \mathcal{M}, \qquad \beta^{(l)} \equiv \begin{pmatrix} 1 & & * \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \bmod \mathcal{M}$$

and hence $\gamma_{k1}^{(l)} \equiv \alpha_k^{-1}\gamma_{k1}(a_k^{-1}b_1 - 1)^l \bmod \mathcal{M}^{r+1}$ for each $l \geqslant 1$. The Engel property of G implies that $\gamma_{k1}^{(l)} \in \mathcal{M}^{r+1}$ for all $l \geqslant 1$. The rest of the argument goes as in the commutative case and shows that all entries of γ are in $\mathcal{M}^{r+1}$. Hence, by induction, they are contained in all powers of $\mathcal{M}$ and we conclude from $\bigcap_{r=0}^{\infty} \mathcal{M}^r = 0$ that γ is the zero matrix. □

Corollary 2.4. *Let R be as in Lemma* 2.3. *Suppose further that an Engel group G of upper-triangular matrices over R contains an element $g = (g_{ij})_{i \leqslant j}$ such that, for some index i, we have $g_{i,i} - g_{i+1,i+1} \in \mathcal{U}(R)$ and $g_{i,i+1} = 0$. Then $x_{i,i+1} = 0$ for all $x = (x_{ij})_{i \leqslant j} \in G$.*

Proof. Notice that the set

$$\left\{ \begin{pmatrix} x_{i,i} & x_{i,i+1} \\ 0 & x_{i+1,i+1} \end{pmatrix} : x \in G \right\}$$

is an Engel group of upper-triangular matrices over R. By the previous lemma, we readily obtain that $x_{i,i+1} = 0$ for all $x \in G$. □

3. The Engel subgroups of upper-triangular matrices

We recall that, for a group G of upper-triangular matrices over R, property (1) is as follows:

$$\begin{aligned}&\text{If for a pair of indices } i<j \text{ there is an element } g'=(g'_{ij})_{i\leqslant j}\\ &\text{in } G \text{ such that } g'_{ii}-g'_{jj}\in\mathcal{U}(R), \text{ then } g_{ij}=0 \text{ for all } g\in G.\end{aligned}$$

Theorem 3.1. *Let G be an Engel subgroup of upper triangular $n\times n$-matrices over a local ring R. Suppose further that R is either commutative or artinian with condition* (2)*. Then G is conjugate in $UT(n,R)$ to a subgroup with property* (1)*.*

Proof. For simplicity we shall say that an upper-triangular matrix $g=(g_{ij})_{i\leqslant j}$ with entries in R satisfies property (x) if

$$g_{ii}-g_{jj}\in\mathcal{U}(R)\quad\Rightarrow\quad g_{ij}=0.$$

Let $\mathcal{M}$ be the (unique) maximal ideal of R and $g=(g_{ij})_{i\leqslant j}$ be an arbitrary fixed element of G. By Lemma 2.1 we may assume, up to conjugacy in $UT(n,R)$, that g satisfies property (x). Suppose that, for some index i, the diagonal entries g_{ii} and $g_{i+1,i+1}$ of g are not congruent modulo $\mathcal{M}$. By property (x) we have that $g_{i,i+1}=0$. Let h be the matrix obtained from the identity $n\times n$-matrix by interchanging its ith and $(i+1)$st rows:

$$h=\begin{pmatrix} I & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I\end{pmatrix}. \tag{4}$$

It is readily seen that conjugation by h transposes the ith and $(i+1)$st rows and also the ith and $(i+1)$st columns. Since $g_{i,i}-g_{i+1,i+1}\in\mathcal{U}(R)$, $g_{i,i+1}=0$ and G is Engel, we know from Corollary 2.4, that $x_{i,i+1}=0$ for all $x\in G$. Hence, all elements of G remain triangular under conjugation by h. Moreover, $h^{-1}gh$ still has property (x) and its (i,i)th and $(i+1,i+1)$st diagonal entries are permuted. Applying this type of conjugation several times if necessary, we see that G is conjugate by an element $\tilde{h}\in GL(n,R)$ to a group $\widetilde{G}$ where g becomes

$$\tilde{g}=\operatorname{diag}(a,b)=\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}, \tag{5}$$

in which a and b are upper-triangular matrices, the diagonal entries of a are pairwise congruent modulo $\mathcal{M}$ and no diagonal entry of a is congruent modulo $\mathcal{M}$ to a diagonal entry of b. The element $\tilde{h}$ is a product of permutation matrices of the form (4). By Lemma 2.3, any x of $\widetilde{G}$ is of the form $x=\operatorname{diag}(\alpha_x,\beta_x)$ where α_x and β_x have the same sizes as a and b, respectively. Let $\widetilde{G}_1=\langle\alpha_x\colon x\in\widetilde{G}\rangle$, $\widetilde{G}_2=\langle\beta_x\colon x\in\widetilde{G}\rangle$. Since $\widetilde{G}_1$ and $\widetilde{G}_2$ are Engel groups of upper-triangular matrices of smaller sizes, induction on n implies that there exists $t_1 \in UT(n_1, R)$ and $t_2 \in UT(n_2, R)$, where n_1 and n_2 are the sizes of a and b, respectively, such that both $t_1^{-1}\widetilde{G}_1 t_1$ and $t_2^{-1}\widetilde{G}_1 t_2$ satisfy (1). Then, taking $t = \operatorname{diag}(t_1, t_2)$, the group $t^{-1}\widetilde{G}t$ obviously verifies (1). Now, let

$$\widetilde{\widetilde{G}} = \tilde{h}t^{-1}\widetilde{G}t\tilde{h}^{-1} = \big(\tilde{h}t\tilde{h}^{-1}\big)^{-1}G\big(\tilde{h}t\tilde{h}^{-1}\big).$$

Because conjugation by $\tilde{h}$ preserves property (1) we see that $\widetilde{\widetilde{G}}$ is upper-triangular and satisfies (1). Hence we will be done if we show that $\tilde{h}t\tilde{h}^{-1} \in UT(n, R)$.

Without loss of generality we may suppose that t_1 and t_2 are transvections. Write $\hat{t}_1 = \operatorname{diag}(t_1, I)$, $\hat{t}_2 = \operatorname{diag}(I, t_2) \in UT(n, R)$ where the I's are identity matrices of appropriate sizes. Then evidently $t = \hat{t}_1\hat{t}_2$. Write also $\tilde{h} = h_{\tau_1}h_{\tau_2}\cdots h_{\tau_m}$, where $h_{\tau_1}, h_{\tau_2}, \ldots$ are permutation matrices of the form (4) and $\tau_1, \tau_2, \ldots$ denote the corresponding permutations (in fact, transpositions) of indices. Observe, in particular, that τ_m is the transposition $(n_1, n_1 + 1)$, since the indices which are involved in a are $1, \ldots, n_1$. Now, notice that when we conjugate an upper-triangular transvection t', say, by one of the permutation matrices h_τ involved in $\tilde{h}$, we obtain again an upper-triangular transvection unless τ interchanges the two indices which determine t'. However, we can avoid this in our case, as shown below.

We remark that conjugation by h_{τ_m} induces a permutation of indices after which we can distinguish two sets which form a partition of the set of all indices: the set $\mathfrak{a}_m$ of those indices coming from $\{1, \ldots, n_1\}$, the set of indices of a, and the set $\mathfrak{b}$ of those coming from the set of indices of b. Moreover $h_{\tau_m}\hat{t}_1h_{\tau_m}^{-1}$ is an upper-triangular transvection whose indices are both in $\mathfrak{a}_m$. In a similar way, $h_{\tau_m}\hat{t}_2h_{\tau_m}^{-1}$ is an upper-triangular transvection whose indices are both in $\mathfrak{b}_m$.

Next, since conjugation by $h_{\tau_{m-1}}$ permutes one index from $\mathfrak{a}_m$ with one from $\mathfrak{b}_m$, we again obtain two disjoint sets of indices $\mathfrak{a}_{m-1}$ and $\mathfrak{b}_{m-1}$, as above, and such that $h_{\tau_{m-1}}h_{\tau_m}\hat{t}_1h_{\tau_m}^{-1}h_{\tau_{m-1}}^{-1}$ and $h_{\tau_{m-1}}h_{\tau_m}\hat{t}_1h_{\tau_m}^{-1}h_{\tau_{m-1}}^{-1}$ are upper-triangular transvections whose indices are in $\mathfrak{a}_{m-1}$ and $\mathfrak{b}_{m-1}$, respectively. Inductively, we obtain that $h\hat{t}_1h^{-1}$ and $h\hat{t}_2h^{-1}$ are both upper triangular transvections and the result follows. $\square$

Denote by $N(n, R)$ the group of all invertible upper-triangular $n \times n$-matrices $g = (g_{ij})_{i \leqslant j}$ with entries in a local ring R such that $g_{ii} \equiv g_{jj} \bmod \mathcal{M}$ for all $1 \leqslant i, j \leqslant n$.

Theorem 3.2. *Let R be a local artinian ring with (unique) maximal ideal $\mathcal{M}$ which satisfies condition (2). For any fixed decomposition $n = n_1 + \cdots + n_s$ the group*

$$G_{n_1,\ldots,n_s} = \begin{pmatrix} N(n_1, R) & & & 0 \\ & N(n_2, R) & & \\ & & \ddots & \\ 0 & & & N(n_s, R) \end{pmatrix}$$

is nilpotent and is a maximal Engel subgroup of $T(n, R)$. Moreover, every maximal Engel subgroup of $T(n, R)$ is conjugate in $GL(n, R)$ to one of the groups $G_{n_1,\ldots,n_s}$.

Proof. Notice that, by Theorem 3.1, every Engel subgroup $G \subseteq T(n,R)$ is conjugate to a group satisfying property (1). Conjugating by permutation matrices of the form (4), we can determine a decomposition $n = n_1 + \cdots + n_s$ such that G is conjugate in $GL(n,R)$ to a subgroup of $G_{n_1,\dots,n_s}$. Thus, it remains to show that $G_{n_1,\dots,n_s}$ is nilpotent.

Evidently, $G_{n_1,\dots,n_s} \simeq N(n_1,R) \times N(n_2,R) \times \cdots \times N(n_s,R)$ and hence it suffices to show that $G = N(n,R)$ is nilpotent ($n \geqslant 1$).

Let m be the nilpotency index of $\mathcal{M}$, that is $\mathcal{M}^m = 0$ and $\mathcal{M}^{m-1} \neq 0$. Observe that (2) implies that the $(k+1)$st term $\Gamma_{k+1}(\mathcal{U}(R))$ of the lower central series of $\mathcal{U}(R)$ is contained in $1 + \mathcal{M}^k$, $k \geqslant 1$, and thus $\mathcal{U}(R)$ is nilpotent of class at most m. Write

$$\mathcal{N}^{(1)} = \begin{pmatrix} 0 & R & R & R & \dots \\ 0 & 0 & R & R & \dots \\ 0 & 0 & 0 & R & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \qquad \mathcal{N}^{(2)} = \begin{pmatrix} 0 & \mathcal{M} & R & R & \dots \\ 0 & 0 & \mathcal{M} & R & \dots \\ 0 & 0 & 0 & \mathcal{M} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

$$\mathcal{N}^{(3)} = \begin{pmatrix} 0 & \mathcal{M}^2 & \mathcal{M} & R & \dots \\ 0 & 0 & \mathcal{M}^2 & \mathcal{M} & \dots \\ 0 & 0 & 0 & \mathcal{M}^2 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \qquad \mathcal{N}^{(4)} = \begin{pmatrix} 0 & \mathcal{M}^3 & \mathcal{M}^2 & \mathcal{M} & \dots \\ 0 & 0 & \mathcal{M}^3 & \mathcal{M}^2 & \dots \\ 0 & 0 & 0 & \mathcal{M}^3 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

etc. As $\mathcal{M}$ is a nilpotent ideal of index m, the last non-zero ideal will be

$$\mathcal{N}^{(n+m-2)} = \begin{pmatrix} 0 & 0 & \dots & 0 & \mathcal{M}^{m-1} \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Denote by D the subgroup of diagonal elements of G. We claim that

$$\begin{aligned} G &\supset \Gamma_2(D) + \mathcal{N}^{(1)} \supset \Gamma_3(D) + \mathcal{N}^{(2)} \supset \cdots \supset \Gamma_m(D) + \mathcal{N}^{(m-1)} \\ &\supset I + \mathcal{N}^{(m)} \supset \cdots \supset I + \mathcal{N}^{(n+m-2)} \supset I \end{aligned} \tag{6}$$

is a central series of G. Indeed, given a diagonal matrix $d = \operatorname{diag}(g_{11},\dots,g_{nn}) \in G$ and a transvection $t = I + e_{ij}(x)$ with $x \in \mathcal{M}^l$ we have that

$$[d,t] = I + e_{ij}\big(g_{ii}^{-1}(g_{ii}x - xg_{jj})\big) \in I + e_{ij}\big(\mathcal{M}^{l+1}\big), \tag{7}$$

as $g_{ii} - g_{jj} \in \mathcal{M}$. Moreover, for any two transvections with $i \neq j'$, one has

$$\big[I + e_{ij}(x), I + e_{i'j'}(y)\big] = I + \delta_{j,i'} e_{ij'}(xy), \tag{8}$$

where $\delta_{j,i'}$ is the Kronecker delta. Notice that, if $j \geqslant j'$ then the right-hand side of (8) is equal to I.

Since G is generated by $UT(n,R)$ and D, it follows in view of (7) and (8) that $[G, I+\mathcal{N}^{(k)}] \subseteq I+\mathcal{N}^{(k+1)}$ for all $k \geqslant 1$. Moreover, for any $h = \mathrm{diag}(h_{11},\dots,h_{nn}) \in \Gamma_{k+1}(D)$ one has $h_{ii} \in \Gamma_{k+1}(\mathcal{U}(R)) \subseteq 1+\mathcal{M}^k$, $i = 1,\dots,n$. Hence taking $t = I + e_{ij}(x)$ with $x \in R$, we see by (7) that $[t,d] \in I + e_{ij}(\mathcal{M}^k)$ and, consequently, $[UT(n,R), \Gamma_{k+1}(D)] \subseteq I+\mathcal{N}^{(k+1)}$. This yields that (6) is a central series and $G = N(n,R)$ is nilpotent. □

Let R be as in Theorem 3.2 and let τ be a partition $\{1,\dots,n\} = \tau_1 \cup \dots \cup \tau_s$ of $\{1,\dots,n\}$ into a disjoint union of subsets. Denote by G_τ the group of all matrices $(g_{ij})_{i\leqslant j} \in T(n,R)$ such that

(i) if $i,j \in \tau_k$ for some $k \in \{1,\dots,s\}$ then $g_{ii} \equiv g_{jj} \bmod \mathcal{M}$;
(ii) $g_{ij} = 0$ for all $i \in \tau_k$ and $j \in \tau_{k'}$ with $k \neq k'$.

Corollary 3.3. *The groups G_τ, with τ running over all partitions of $\{1,\dots,n\}$ into disjoint union of subsets, are nilpotent, pairwise non-conjugate in $T(n,R)$ and every maximal Engel subgroup of $T(n,R)$ is conjugate to one of them. Moreover, all the elements in the conjugacy class of G_τ are obtained conjugating G_τ by the elements of $UT(n,R)$.*

Proof. By Theorem 3.1 every Engel subgroup of $T(n,R)$ satisfies property (1) and hence, up to conjugacy in $UT(n,R)$, is a subgroup of G_τ for some partition τ. It follows from the proof of Theorem 3.1 that G_τ is conjugate in $GL(n,R)$ to $G_{n_1,\dots,n_s}$ where n_i is the number of elements in τ_i $(i = 1,\dots,s)$. Hence G_τ is maximal Engel and nilpotent in view of Theorem 3.2. □

From Theorem 3.2 we readily obtain the following.

Corollary 3.4. *The groups $G_{n_1,n_2,\dots}$ with $n = n_1+n_2+\cdots$ running over all decompositions such that $n_1 \geqslant n_2 \geqslant \cdots$, form a full set of representatives of the isomorphism classes of the maximal Engel subgroups of $T(n,R)$.*

The length of the central series (6) of the group $G = N(n,R)$ is $n+m-1$, so we wonder whether or not it is the nilpotency class of G. We examine this by analyzing the lower central series of $N(n,R)$. It was seen above that $[G, I+\mathcal{N}^{(k)}] \subseteq I+\mathcal{N}^{(k+1)}$. We have that

$$\bigl[D, I+e_{ij}\bigl(\mathcal{M}^l\bigr)\bigr] = I+e_{ij}\bigl(\mathcal{M}^{(l+1)}\bigr) \tag{9}$$

for each $l \geqslant 0$. In fact, take $I+e_{ij}(z)$ with $z \in \mathcal{M}^{l+1}$. Write $z = \sum xy$ where $x \in \mathcal{M}^l$, $y \in \mathcal{M}$. Then $d = \mathrm{diag}(1,\dots,1,1-y,1,\dots,1) \in D$ with $1-y$ placed in the jth position. By (7) we have that $[d, I+e_{ij}(x)] = I+e_{ij}(xy)$ and thus $I+e_{ij}(z) = \prod I+e_{ij}(xy) \in [D, I+e_{ij}(\mathcal{M}^l)]$ which means that $[D, I+e_{ij}(\mathcal{M}^l)] \supseteq I+e_{ij}(\mathcal{M}^{l+1})$, proving (9).

Using formulas (8) and (9), it follows that the commutator subgroup of $N(n,R)$ is

$$\bigl[D\bigl(I+\mathcal{N}^{(1)}\bigr), D\bigl(I+\mathcal{N}^{(1)}\bigr)\bigr] = \Gamma_2(D)\bigl(I+\mathcal{N}^{(2)}\bigr).$$

Next for $g = \operatorname{diag}(g_{11}, \dots, g_{nn}) \in \Gamma_2(D)$ observe that $g_{ii} - g_{jj} \in \mathcal{M}^2$ for any $1 \leqslant i, j \leqslant n$. Indeed, it will suffice to prove our claim in the case when g is a generator of $\Gamma_2(D)$ and thus $g_{ii} = [h_{ii}, f_{ii}]$ and $g_{jj} = [h_{jj}, f_{jj}]$ for some $h_{ii}, f_{ii}, h_{jj}, f_{jj} \in \mathcal{U}(R)$. We know that $h_{jj} = h_{ii}(1+x)$ and $f_{jj} = f_{ii}(1+y)$ with $x, y \in \mathcal{M}$. Then

$$\begin{aligned} g_{jj} &= \bigl[h_{ii}(1+x), f_{ii}(1+y)\bigr] \\ &= \bigl([h_{ii}, 1+y][h_{ii}, f_{ii}]^{(1+y)}\bigr)^{(1+x)}[1+x, 1+y][1+x, f_{ii}]^{(1+y)}. \end{aligned}$$

Since $\mathcal{M}$ is central modulo $\mathcal{M}^2$, it follows that $g_{jj} \in [h_{ii}, f_{ii}](1+\mathcal{M}^2)$ and consequently $g_{ii} - g_{jj} \in \mathcal{M}^2$ as desired.

We see now that if $g = \operatorname{diag}(g_{11}, \dots, g_{nn}) \in \Gamma_2(D)$, then for any $I + e_{ij}(z) \in I + \mathcal{N}^{(1)}$ we have by (7) that

$$\bigl[I + e_{ij}(z), g\bigr] \subseteq I + e_{ij}\bigl(\mathcal{M}^2\bigr) \subseteq I + \mathcal{N}^{(3)},$$

so that

$$\bigl[I + \mathcal{N}^{(1)}, \Gamma_2(D)\bigr] \subseteq I + \mathcal{N}^{(3)}$$

and thus, in view of formula (9), the third term of the lower central series of $N(n, R)$ is

$$\bigl[D\bigl(I + \mathcal{N}^{(1)}\bigr), \Gamma_2(D)\bigl(I + \mathcal{N}^{(2)}\bigr)\bigr] = \Gamma_3(D)\bigl(I + \mathcal{N}^{(3)}\bigr).$$

Is it true in general that $g_{ii} - g_{jj} \in \mathcal{M}^k$ for any $1 \leqslant i, j \leqslant n$ if $g = \operatorname{diag}(g_{11}, \dots, g_{nn}) \in \Gamma_k(D)$ and $k \geqslant 1$? If we assume this for $k-1$, we may write $g_{ii} = [h_{ii}, f_{ii}]$, $g_{jj} = [h_{jj}, f_{jj}]$, $h_{jj} = h_{ii}(1+x)$, $f_{jj} = f_{ii}(1+y)$ with $h_{ii} \in \Gamma_{k-1}(\mathcal{U}(R))$, $f_{ii} \in \mathcal{U}(R)$, $x \in \mathcal{M}^{k-1}$ and $y \in \mathcal{M}$. Then, since clearly

$$[1+x, 1+y][1+x, f_{ii}]^{(1+y)} \in 1 + \mathcal{M}^k,$$

the above commutator calculation shows that $g_{jj} \in g_{ii}(1+\mathcal{M}^k)$ if and only if $[h_{ii}, 1+y][h_{ii}, f_{ii}]^{(1+y)} \in g_{ii}(1+\mathcal{M}^k)$. Since $[h_{ii}, f_{ii}] = g_{ii} \in 1 + \mathcal{M}^{k-1}$, we have $[g_{ii}, 1+y] \in 1 + \mathcal{M}^k$ and thus

$$g_{ii} - g_{jj} \in \mathcal{M}^k \quad\Longleftrightarrow\quad [h_{ii}, 1+y] \in 1 + \mathcal{M}^k.$$

We conclude from the above argument that if $[\Gamma_{l-1}(\mathcal{U}(R)), 1+\mathcal{M}] \subseteq 1 + \mathcal{M}^l$ for all $2 \leqslant l \leqslant k$ then for each such l and any $g = \operatorname{diag}(g_{11}, \dots, g_{nn}) \in \Gamma_l(D)$ one has $g_{ii} - g_{jj} \in \mathcal{M}^l$ for all $1 \leqslant i, j \leqslant n$. Applying (7), again we see that $[I + \mathcal{N}^{(1)}, \Gamma_l(D)] \subseteq I + \mathcal{N}^{(l+1)}$ which yields that the $(l+1)$st term of the lower central series of $N(n, R)$ is

$$\bigl[D\bigl(I + \mathcal{N}^{(1)}\bigr), \Gamma_l(D)\bigl(I + \mathcal{N}^{(l)}\bigr)\bigr] = \Gamma_{l+1}(D)\bigl(I + \mathcal{N}^{(l+1)}\bigr), \tag{10}$$

$2 \leqslant l \leqslant k$.

Thus we have shown that if $[\Gamma_{k-1}(\mathcal{U}(R)), 1+\mathcal{M}] \subseteq 1+\mathcal{M}^k$ does hold for all k, then all terms of the lower central series of $N(n,R)$ are determined by (10) and the nilpotency class of $N(n,R)$ is $n+m-2$. This obviously happens if $\Gamma_2(\mathcal{U}(R)) \subseteq 1+\mathcal{M}^2$ as, by induction, $[\Gamma_{k-1}(\mathcal{U}(R)), 1+\mathcal{M}] \subseteq [1+\mathcal{M}^{k-1}, 1+\mathcal{M}] \subseteq 1+\mathcal{M}^k$ for all k.

On the other hand, this is true also if $[\mathcal{M},\mathcal{M}]_{\mathrm{Lie}} \subseteq \mathcal{M}^3$. Indeed, induction shows that $[\mathcal{M}^{k-2},\mathcal{M}]_{\mathrm{Lie}} \subseteq \mathcal{M}^k$. Since $\Gamma_{k-1}(\mathcal{U}(R)) \subseteq 1+\mathcal{M}^{k-2}$, it follows again that $[1+\mathcal{M}^{k-1}, 1+\mathcal{M}] \subseteq 1+\mathcal{M}^k$ for all k.

Suppose that $[\Gamma_{k-1}(\mathcal{U}(R)), 1+\mathcal{M}] \subseteq 1+\mathcal{M}^k$ does not hold for all k and let k_0 be the least integer with $[\Gamma_{k_0}(\mathcal{U}(R)), 1+\mathcal{M}] \nsubseteq 1+\mathcal{M}^{k_0+1}$. Evidently $2 \leqslant k_0 \leqslant m-1$. It follows from the above consideration that if $h \in \Gamma_{k_0}(\mathcal{U}(R))$ and $y \in \mathcal{M}$ are such that $[h, 1+y] \notin 1+\mathcal{M}^{k_0+1}$, then taking $h' = \mathrm{diag}(h,\dots,h)$ and $f = \mathrm{diag}(1, 1+y, 1, \dots)$, we see that $g = [h', f] = \mathrm{diag}(1, [h, 1+y], 1, \dots, 1) \in \Gamma_{k_0+1}(D)$ and by (7),

$$\bigl[g, I+e_{12}(1)\bigr] = I+e_{12}(v), \quad \text{with } v = 1-[h,1+y] \in \mathcal{M}^{k_0} \setminus \mathcal{M}^{k_0+1}.$$

Thus $\widetilde{\mathcal{Z}} = [I+\mathcal{N}^{(1)}, \Gamma_{k_0+1}(D)] \nsubseteq I+\mathcal{N}^{(k_0+2)}$. Since $\Gamma_{k_0+1}(\mathcal{U}(R)) \subseteq 1+\mathcal{M}^{k_0}$, one has that $\widetilde{\mathcal{Z}} \subseteq I+\mathcal{N}^{(k_0+1)}$. Then the (k_0+2)nd term of the lower central series of $N(n,R)$ is

$$\begin{aligned}\bigl[D\bigl(I+\mathcal{N}^{(1)}\bigr), \Gamma_{k_0+1}(D)\bigl(I+\mathcal{N}^{(k_0+1)}\bigr)\bigr] &= \Gamma_{k_0+2}(D)\widetilde{\mathcal{Z}}\bigl(I+\mathcal{N}^{(k_0+2)}\bigr)\\ &\supsetneq \Gamma_{k_0+2}(D)\bigl(I+\mathcal{N}^{(k_0+2)}\bigr).\end{aligned}$$

Suppose that $k_0 = m-1$. If $n=2$, then $k_0+2 = m-1+2 = n+m-1$ and the above calculated term $\Gamma_{k_0+2}(N(n,R))$ is the last non-identity term of the lower central series of $N(n,R)$. If $n>2$, taking an element from $\widetilde{\mathcal{Z}}$ which is not contained in $I+\mathcal{N}^{(k_0+2)}$, we see using (8) that $[I+\mathcal{N}^{(1)}, \widetilde{\mathcal{Z}}]$ has a nonidentity element from $I+e_{13}(\mathcal{M}^{m-1})$. Since this element is not contained in $I+\mathcal{N}^{(k_0+3)}$, the (k_0+3)rd term of the lower central series of $N(n,R)$ strictly contains $\Gamma_{k_0+3}(D)(I+\mathcal{N}^{(k_0+3)})$. We see by induction that the $(n+m-1)$st term of the lower central series of $N(n,R)$ contains a non-identity element from $I+e_{1n}(\mathcal{M}^{m-1})$. We have shown thus that if $k_0 = m-1$, then the nilpotency class of $N(n,R)$ is $n+m-1$.

If $k_0 < m-1$, the nilpotenthy class of $N(n,R)$ still depends on the properties of R. However, for all $l \geqslant 2$ the lth term of the lower central series of $N(n,R)$ contains $\Gamma_l(D)(I+\mathcal{N}^{(l)})$ and is contained in $\Gamma_l(D)(I+\mathcal{N}^{(l-1)})$ so that the nilpotenthy class is either $n+m-2$ or $n+m-1$. Assuming a rather natural condition on R we can guarantee that it will be $n+m-1$. The condition is

$$v \in \mathcal{M}, \quad v\mathcal{M} \subseteq \mathcal{M}^s \quad \Rightarrow \quad v \in \mathcal{M}^{s-1}, \quad \text{for all } s \geqslant 2. \tag{11}$$

Assuming (11) and taking the above considered element $v = 1-[h,1+y]$, we see that there exists $z \in \mathcal{M}$ with $vz \notin \mathcal{M}^{k_0+2}$. Then by (7) one has $[\mathrm{diag}(1, 1-z, 1, \dots, 1), I+e_{12}(v)] = I+e_{12}(vz)$ and the (k_0+3)rd term of the lower central series of $N(n,R)$ strictly contains $\Gamma_{k_0+3}(D)(I+\mathcal{N}^{(k_0+3)})$. Using (11) in this way, we produce by induction an element $I+e_{12}(w)$ of the $(m+1)$st term of the lower central series of $N(n,R)$ which is not contained in $I+\mathcal{N}^{(m+1)}$. Then applying (8) several times, as in case $k_0 = m-1$, we obtain in the $(n+m-1)$st term of the lower central series of $N(n,R)$ a nonidentity element from $I+e_{1n}(\mathcal{M}^{m-1})$. This means that the nilpotenthy class is in fact $n+m-1$.

We have obtained the following result.

Proposition 3.5. *Let R be as in Theorem* 3.2 *and $n \geqslant 2$.*

(i) *If $[\Gamma_{k-1}(\mathcal{U}(R)), 1+\mathcal{M}] \subseteq 1+\mathcal{M}^k$ for all k, then the lower central series of $N(n,R)$ is*

$$G \supset \Gamma_2(D)+\mathcal{N}^{(2)} \supset \Gamma_3(D)+\mathcal{N}^{(3)} \supset \cdots \supset \Gamma_m(D)+\mathcal{N}^{(m)}$$
$$\supset I+\mathcal{N}^{(m+1)} \supset \cdots \supset I+\mathcal{N}^{(n+m-2)} \supset I,$$

and the nilpotency class of $N(n,R)$ is $n+m-2$. This happens if either $\Gamma_2(\mathcal{U}(R)) \subseteq 1+\mathcal{M}^2$ or if $[\mathcal{M},\mathcal{M}]_{\mathrm{Lie}} \subseteq \mathcal{M}^3$.

(ii) *If k_0 is the least integer with $[\Gamma_{k_0}(\mathcal{U}(R)), 1+\mathcal{M}] \nsubseteq 1+\mathcal{M}^{k_0+1}$, then $2 \leqslant k_0 \leqslant m-1$, the first k_0+1 terms of the lower central series of $N(n,R)$ are as in item* (i) *and for each $k_0+2 \leqslant l \leqslant n+m-1$ the lth term $\Gamma_l(N(n,R))$ of the lower central series of $N(n,R)$ satisfies*

$$\Gamma_l(D)\bigl(I+\mathcal{N}^{(l)}\bigr) \subseteq \Gamma_l\bigl(N(n,R)\bigr) \subseteq \Gamma_l(D)\bigl(I+\mathcal{N}^{(l-1)}\bigr).$$

In particular, the nilpotency class of $N(n,R)$ is either $n+m-2$ or $n+m-1$. The latter occurs whenever R satisfies (11) *or $k_0 = m-1$.*

The next corollary immediately follows from item (i) of Proposition 3.5.

Corollary 3.6. *If R is a commutative local artinian ring and $n \geqslant 2$, then $N(n,R)$ is nilpotent of class $n+m-2$.*

On the other hand, since $2 \leqslant k_0 \leqslant m-1$, item (ii) of Proposition 3.5 does not occur if $m=2$. Thus we immediately have:

Corollary 3.7. *Let R be as in Theorem* 3.2 *and $n \geqslant 2$. If $\mathcal{M} \neq 0$ and $\mathcal{M}^2 = 0$, then $N(n,R)$ is nilpotent of class $n+m-2=n$.*

If R is a K-algebra satisfying the conditions of Theorem 3.2 and $R = K \oplus \mathcal{M}$ as vector spaces, then $\Gamma_2(\mathcal{U}(R)) = \Gamma_2(1+\mathcal{M}) \subseteq 1+\mathcal{M}^2$, so that R verifies item (i) of Proposition 3.5. An example of such a ring is given by the group algebra KG of a finite p-group G over a field K of characteristic p. Indeed, it was observed already in the Introduction that KG is local with $\mathcal{M} = \Delta(G)$ which satisfies (2). Moreover, evidently $KG = K \oplus \Delta(G)$. Hence we obtain the following corollary.

Corollary 3.8. *Let R be as in Theorem* 3.2 *and $n \geqslant 2$. If R is a K-algebra and $R = K \oplus \mathcal{M}$ as vector spaces, then $N(n,R)$ is nilpotent of class $n+m-2$. In particular, this occurs if $R = KG$, the group algebra of a finite p-group G over a field K of characteristic p.*

Acknowledgment

The first author is grateful to Professor Arnaldo Mandel for useful conversations.

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